11^{th} International Mathematical Competition for University Students
Skopje, 25–26 July 2004

Solutions for problems on Day 1

Problem 1. Let S be an infinite set of real numbers such that $|s_1 + s_2 + \cdots + s_k| < 1$ for every finite subset $\{s_1, s_2, \ldots, s_k\} \subset S$. Show that S is countable. [20 points]

Solution. Let $S_n = S \cap (\frac{1}{n}, \infty)$ for any integer $n > 0$. It follows from the inequality that $|S_n| < n$. Similarly, if we define $S_{-n} = S \cap (-\infty, -\frac{1}{n})$, then $|S_{-n}| < n$. Any nonzero $x \in S$ is an element of some S_n or S_{-n}, because there exists an n such that $x > \frac{1}{n}$, or $x < -\frac{1}{n}$. Then $S \subset \{0\} \cup \bigcup_{n\in N} (S_n \cup S_{-n})$, S is a countable union of finite sets, and hence countable.

Problem 2. Let $P(x) = x^2 - 1$. How many distinct real solutions does the following equation have:

$$\underbrace{P(P(\ldots(P}_{2004}(x))\ldots)) = 0 \ ? \qquad \text{[20 points]}$$

Solution. Put $P_n(x) = \underbrace{P(P(...(P}_{n}(x))...))$. As $P_1(x) \geq -1$, for each $x \in R$, it must be that $P_{n+1}(x) = P_1(P_n(x)) \geq -1$, for each $n \in N$ and each $x \in R$. Therefore the equation $P_n(x) = a$, where $a < -1$ has no real solutions. Let us prove that the equation $P_n(x) = a$, where $a > 0$, has exactly two distinct real solutions. To this end we use mathematical induction by n. If $n = 1$ the assertion follows directly. Assuming that the assertion holds for a $n \in N$ we prove that it must also hold for $n + 1$. Since $P_{n+1}(x) = a$ is equivalent to $P_1(P_n(x)) = a$, we conclude that $P_n(x) = \sqrt{a+1}$ or $P_n(x) = -\sqrt{a+1}$. The equation $P_n(x) = \sqrt{a+1}$, as $\sqrt{a+1} > 1$, has exactly two distinct real solutions by the inductive hypothesis, while the equation $P_n(x) = -\sqrt{a+1}$ has no real solutions (because $-\sqrt{a+1} < -1$). Hence the equation $P_{n+1}(x) = a$, has exactly two distinct real solutions.

Let us prove now that the equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions. Again we use mathematical induction. If $n = 1$ the solutions are $x = \pm 1$, and if $n = 2$ the solutions are $x = 0$ and $x = \pm\sqrt{2}$, so in both cases the number of solutions is equal to $n + 1$. Suppose that the assertion holds for some $n \in N$. Note that $P_{n+2}(x) = P_2(P_n(x)) = P_n^2(x)(P_n^2(x) - 2)$, so the set of all real solutions of the equation $P_{n+2} = 0$ is exactly the union of the sets of all real solutions of the equations $P_n(x) = 0$, $P_n(x) = \sqrt{2}$ and $P_n(x) = -\sqrt{2}$. By the inductive hypothesis the equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions, while the equations $P_n(x) = \sqrt{2}$ and $P_n(x) = -\sqrt{2}$ have two and no distinct real solutions, respectively. Hence, the sets above being pairwise disjoint, the equation $P_{n+2}(x) = 0$ has exactly $n + 3$ distinct real solutions. Thus we have proved that, for each $n \in N$, the equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions, so the answer to the question posed in this problem is 2005.

Problem 3. Let S_n be the set of all sums $\sum_{k=1}^{n} x_k$, where $n \geq 2$, $0 \leq x_1, x_2, \ldots, x_n \leq \frac{\pi}{2}$ and

$$\sum_{k=1}^{n} \sin x_k = 1\,.$$

a) Show that S_n is an interval. [10 points]
b) Let l_n be the length of S_n. Find $\lim_{n\to\infty} l_n$. [10 points]

Solution. (a) Equivalently, we consider the set

$$Y = \{y = (y_1, y_2, ..., y_n)|\ 0 \leq y_1, y_2, ..., y_n \leq 1,\ y_1 + y_2 + ... + y_n = 1\} \subset R^n$$

and the image $f(Y)$ of Y under

$$f(y) = \arcsin y_1 + \arcsin y_2 + ... + \arcsin y_n.$$

Note that $f(Y) = S_n$. Since Y is a connected subspace of R^n and f is a continuous function, the image $f(Y)$ is also connected, and we know that the only connected subspaces of R are intervals. Thus S_n is an interval.

(b) We prove that

$$n\ \arcsin\frac{1}{n} \leq x_1 + x_2 + ... + x_n \leq \frac{\pi}{2}.$$

Since the graph of $\sin x$ is concave down for $x \in [0, \frac{\pi}{2}]$, the chord joining the points $(0, 0)$ and $(\frac{\pi}{2}, 1)$ lies below the graph. Hence

$$\frac{2x}{\pi} \leq \sin x \text{ for all } x \in [0, \frac{\pi}{2}]$$

and we can deduce the right-hand side of the claim:

$$\frac{2}{\pi}(x_1 + x_2 + ... + x_n) \leq \sin x_1 + \sin x_2 + ... + \sin x_n = 1.$$

The value 1 can be reached choosing $x_1 = \frac{\pi}{2}$ and $x_2 = \cdots = x_n = 0$.

The left-hand side follows immediately from Jensen's inequality, since $\sin x$ is concave down for $x \in [0, \frac{\pi}{2}]$ and $0 \leq \frac{x_1+x_2+...+x_n}{n} < \frac{\pi}{2}$

$$\frac{1}{n} = \frac{\sin x_1 + \sin x_2 + ... + \sin x_n}{n} \leq \sin\frac{x_1 + x_2 + ... + x_n}{n}.$$

Equality holds if $x_1 = \cdots = x_n = \arcsin\frac{1}{n}$.

Now we have computed the minimum and maximum of interval S_n; we can conclude that $S_n = [n\ \arcsin\frac{1}{n}, \frac{\pi}{2}]$. Thus $l_n = \frac{\pi}{2} - n\ \arcsin\frac{1}{n}$ and

$$\lim_{n\to\infty} l_n = \frac{\pi}{2} - \lim_{n\to\infty}\frac{\arcsin(1/n)}{1/n} = \frac{\pi}{2} - 1.$$

Problem 4. Suppose $n \geq 4$ and let M be a finite set of n points in $\mathbb{R}^3$, no four of which lie in a plane. Assume that the points can be coloured black or white so that any sphere which intersects M in at least four points has the property that exactly half of the points in the intersection of M and the sphere are white. Prove that all of the points in M lie on one sphere. [20 points]

Solution. Define $f : M \to \{-1, 1\}$, $f(X) = \begin{cases} -1, \text{ if } X \text{ is white} \\ 1, \text{ if } X \text{ is black} \end{cases}$. The given condition becomes $\sum_{X\in S} f(X) = 0$ for any sphere S which passes through at least 4 points of M. For any 3 given points A, B, C in M, denote by $S(A, B, C)$ the set of all spheres which pass through A, B, C and at least one other point of M and by $|S(A, B, C)|$ the number of these spheres. Also, denote by $\sum$ the sum $\sum_{X\in M} f(X)$.

We have

$$0 = \sum_{S\in S(A,B,C)}\sum_{X\in S} f(X) = (|S(A, B, C)| - 1)(f(A) + f(B) + f(C)) + \sum \tag{1}$$

since the values of A, B, C appear $|S(A, B, C)|$ times each and the other values appear only once.

If there are 3 points A, B, C such that $|S(A, B, C)| = 1$, the proof is finished.

If $|S(A, B, C)| > 1$ for any distinct points A, B, C in M, we will prove at first that $\sum = 0$.

Assume that $\sum > 0$. From (1) it follows that $f(A) + f(B) + f(C) < 0$ and summing by all $\binom{n}{3}$ possible choices of (A, B, C) we obtain that $\binom{n}{3}\sum < 0$, which means $\sum < 0$ (contradicts the starting assumption). The same reasoning is applied when assuming $\sum < 0$.

Now, from $\sum = 0$ and (1), it follows that $f(A) + f(B) + f(C) = 0$ for any distinct points A, B, C in M. Taking another point $D \in M$, the following equalities take place

$$f(A) + f(B) + f(C) = 0$$
$$f(A) + f(B) + f(D) = 0$$
$$f(A) + f(C) + f(D) = 0$$
$$f(B) + f(C) + f(D) = 0$$

which easily leads to $f(A) = f(B) = f(C) = f(D) = 0$, which contradicts the definition of f.

Problem 5. Let X be a set of $\binom{2k-4}{k-2} + 1$ real numbers, $k \geq 2$. Prove that there exists a monotone sequence $\{x_i\}_{i=1}^k \subseteq X$ such that

$$|x_{i+1} - x_1| \geq 2|x_i - x_1|$$

for all $i = 2, \ldots, k - 1$. [20 points]

Solution. We prove a more general statement:

Lemma. Let $k, l \geq 2$, let X be a set of $\binom{k+l-4}{k-2}+1$ real numbers. Then either X contains an increasing sequence $\{x_i\}_{i=1}^k \subseteq X$ of length k and

$$|x_{i+1}-x_1| \geq 2|x_i - x_1| \quad \forall i = 2,\dots,k-1,$$

or X contains a decreasing sequence $\{x_i\}_{i=1}^l \subseteq X$ of length l and

$$|x_{i+1}-x_1| \geq 2|x_i - x_1| \quad \forall i = 2,\dots,l-1.$$

Proof of the lemma. We use induction on $k+l$. In case $k=2$ or $l=2$ the lemma is obviously true.

Now let us make the induction step. Let m be the minimal element of X, M be its maximal element. Let

$$X_m = \{x \in X : x \leq \frac{m+M}{2}\}, \quad X_M = \{x \in X : x > \frac{m+M}{2}\}.$$

Since $\binom{k+l-4}{k-2} = \binom{k+(l-1)-4}{k-2} + \binom{(k-1)+l-4}{(k-1)-2}$, we can see that either

$$|X_m| \geq \binom{(k-1)+l-4}{(k-1)-2}+1, \quad \text{or} \quad |X_M| \geq \binom{k+(l-1)-4}{k-2}+1.$$

In the first case we apply the inductive assumption to X_m and either obtain a decreasing sequence of length l with the required properties (in this case the inductive step is made), or obtain an increasing sequence $\{x_i\}_{i=1}^{k-1} \subseteq X_m$ of length $k-1$. Then we note that the sequence $\{x_1, x_2, \dots, x_{k-1}, M\} \subseteq X$ has length k and all the required properties.

In the case $|X_M| \geq \binom{k+(l-1)-4}{k-2}+1$ the inductive step is made in a similar way. Thus the lemma is proved.

The reader may check that the number $\binom{k+l-4}{k-2}+1$ cannot be smaller in the lemma.

Problem 6. For every complex number $z \notin \{0,1\}$ define

$$f(z) := \sum (\log z)^{-4},$$

where the sum is over all branches of the complex logarithm.

a) Show that there are two polynomials P and Q such that $f(z) = P(z)/Q(z)$ for all $z \in \mathbb{C} \setminus \{0,1\}$. [10 points]

b) Show that for all $z \in \mathbb{C} \setminus \{0,1\}$

$$f(z) = z\frac{z^2+4z+1}{6(z-1)^4}. \qquad \text{[10 points]}$$

Solution 1. It is clear that the left hand side is well defined and independent of the order of summation, because we have a sum of the type $\sum n^{-4}$, and the branches of the logarithms do not matter because all branches are taken. It is easy to check that the convergence is locally uniform on $\mathbb{C} \setminus \{0,1\}$; therefore, f is a holomorphic function on the complex plane, except possibly for isolated singularities at 0 and 1. (We omit the detailed estimates here.)

The function log has its only (simple) zero at $z=1$, so f has a quadruple pole at $z=1$.

Now we investigate the behavior near infinity. We have $\mathrm{Re}(\log(z)) = \log|z|$, hence (with $c := \log|z|$)

$$\begin{aligned}
|\sum (\log z)^{-4}| &\leq \sum |\log z|^{-4} = \sum (\log|z| + 2\pi i n)^{-4} + O(1) \\
&= \int_{-\infty}^{\infty} (c + 2\pi i x)^{-4}\, dx + O(1) \\
&= c^{-4} \int_{-\infty}^{\infty} (1 + 2\pi i x/c)^{-4}\, dx + O(1) \\
&= c^{-3} \int_{-\infty}^{\infty} (1 + 2\pi i t)^{-4}\, dt + O(1) \\
&\leq \alpha(\log|z|)^{-3}
\end{aligned}$$

for a universal constant α. Therefore, the infinite sum tends to 0 as $|z| \to \infty$. In particular, the isolated singularity at ∞ is not essential, but rather has (at least a single) zero at ∞.

The remaining singularity is at $z=0$. It is readily verified that $f(1/z)=f(z)$ (because $\log(1/z)=-\log(z)$); this implies that f has a zero at $z=0$.

We conclude that the infinite sum is holomorphic on $\mathbb{C}$ with at most one pole and without an essential singularity at ∞, so it is a rational function, i.e. we can write $f(z)=P(z)/Q(z)$ for some polynomials P and Q which we may as well assume coprime. This solves the first part.

Since f has a quadruple pole at $z=1$ and no other poles, we have $Q(z)=(z-1)^4$ up to a constant factor which we can as well set equal to 1, and this determines P uniquely. Since $f(z)\to 0$ as $z\to\infty$, the degree of P is at most 3, and since $P(0)=0$, it follows that $P(z)=z(az^2+bz+c)$ for yet undetermined complex constants a,b,c.

There are a number of ways to compute the coefficients a,b,c, which turn out to be $a=c=1/6$, $b=2/3$. Since $f(z)=f(1/z)$, it follows easily that $a=c$. Moreover, the fact $\lim\limits_{z\to 1}(z-1)^4 f(z)=1$ implies $a+b+c=1$ (this fact follows from the observation that at $z=1$, all summands cancel pairwise, except the principal branch which contributes a quadruple pole). Finally, we can calculate

$$f(-1)=\pi^{-4}\sum_{n odd} n^{-4}=2\pi^{-4}\sum_{n\geq 1 odd} n^{-4}=2\pi^{-4}\left(\sum_{n\geq 1} n^{-4}-\sum_{n\geq 1 even} n^{-4}\right)=\frac{1}{48}.$$

This implies $a-b+c=-1/3$. These three equations easily yield a,b,c.

Moreover, the function f satisfies $f(z)+f(-z)=16f(z^2)$: this follows because the branches of $\log(z^2)=\log((-z)^2)$ are the numbers $2\log(z)$ and $2\log(-z)$. This observation supplies the two equations $b=4a$ and $a=c$, which can be used instead of some of the considerations above.

Another way is to compute $g(z)=\sum\frac{1}{(\log z)^2}$ first. In the same way, $g(z)=\frac{dz}{(z-1)^2}$. The unknown coefficient d can be computed from $\lim\limits_{z\to 1}(z-1)^2 g(z)=1$; it is $d=1$. Then the exponent 2 in the denominator can be increased by taking derivatives (see Solution 2). Similarly, one can start with exponent 3 directly.

A more straightforward, though tedious way to find the constants is computing the first four terms of the Laurent series of f around $z=1$. For that branch of the logarithm which vanishes at 1, for all $|w|<\frac{1}{2}$ we have

$$\log(1+w)=w-\frac{w^2}{2}+\frac{w^3}{3}-\frac{w^4}{4}+O(|w|^5);$$

after some computation, one can obtain

$$\frac{1}{\log(1+w)^4}=w^{-4}+2w^{-2}+\frac{7}{6}w^{-2}+\frac{1}{6}w^{-1}+O(1).$$

The remaining branches of logarithm give a bounded function. So

$$f(1+w)=w^{-4}+2w^{-2}+\frac{7}{6}w^{-2}+\frac{1}{6}w^{-1}$$

(the remainder vanishes) and

$$f(z)=\frac{1+2(z-1)+\frac{7}{6}(z-1)^2+\frac{1}{6}(z-1)^3}{(z-1)^4}=\frac{z(z^2+4z+1)}{6(z-1)^4}.$$

Solution 2. ¿From the well-known series for the cotangent function,

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{w+2\pi i\cdot k}=\frac{i}{2}\cot\frac{iw}{2}$$

and

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{\log z+2\pi i\cdot k}=\frac{i}{2}\cot\frac{i\log z}{2}=\frac{i}{2}\cdot i\frac{e^{2i\cdot\frac{i\log z}{2}}+1}{e^{2i\cdot\frac{i\log z}{2}}-1}=\frac{1}{2}+\frac{1}{z-1}.$$

Taking derivatives we obtain

$$\sum\frac{1}{(\log z)^2}=-z\cdot\left(\frac{1}{2}+\frac{1}{z-1}\right)'=\frac{z}{(z-1)^2},$$

$$\sum\frac{1}{(\log z)^3}=-\frac{z}{2}\cdot\left(\frac{z}{(z-1)^2}\right)'=\frac{z(z+1)}{2(z-1)^3}$$

and

$$\sum\frac{1}{(\log z)^4}=-\frac{z}{3}\cdot\left(\frac{z(z+1)}{2(z-1)^3}\right)'=\frac{z(z^2+4z+1)}{2(z-1)^4}.$$