13^{th} International Mathematics Competition for University Students
Odessa, July 20-26, 2006
First Day

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements.
(a) If f is continuous and $\operatorname{range}(f) = \mathbb{R}$ then f is monotonic.
(b) If f is monotonic and $\operatorname{range}(f) = \mathbb{R}$ then f is continuous.
(c) If f is monotonic and f is continuous then $\operatorname{range}(f) = \mathbb{R}$.
(20 points)

Solution. (a) False. Consider function $f(x) = x^3 - x$. It is continuous, $\operatorname{range}(f) = \mathbb{R}$ but, for example, $f(0) = 0$, $f(\frac{1}{2}) = -\frac{3}{8}$ and $f(1) = 0$, therefore $f(0) > f(\frac{1}{2})$, $f(\frac{1}{2}) < f(1)$ and f is not monotonic.

(b) True. Assume first that f is non-decreasing. For an arbitrary number a, the limits $\lim\limits_{a-} f$ and $\lim\limits_{a+} f$ exist and $\lim\limits_{a-} f \le \lim\limits_{a+} f$. If the two limits are equal, the function is continuous at a. Otherwise, if $\lim\limits_{a-} f = b < \lim\limits_{a+} f = c$, we have $f(x) \le b$ for all $x < a$ and $f(x) \ge c$ for all $x > a$; therefore $\operatorname{range}(f) \subset (-\infty, b) \cup (c, \infty) \cup \{f(a)\}$ cannot be the complete $\mathbb{R}$.

For non-increasing f the same can be applied writing reverse relations or $g(x) = -f(x)$.

(c) False. The function $g(x) = \arctan x$ is monotonic and continuous, but $\operatorname{range}(g) = (-\pi/2, \pi/2) \ne \mathbb{R}$.

Problem 2. Find the number of positive integers x satisfying the following two conditions:

1. $x < 10^{2006}$;
2. $x^2 - x$ is divisible by 10^{2006}.

(20 points)

Solution 1. Let $S_k = \left\{0 < x < 10^k \,\middle|\, x^2 - x \text{ is divisible by } 10^k\right\}$ and $s(k) = |S_k|, k \ge 1$. Let $x = a_{k+1}a_k \dots a_1$ be the decimal writing of an integer $x \in S_{k+1}, k \ge 1$. Then obviously $y = a_k \dots a_1 \in S_k$. Now, let $y = a_k \dots a_1 \in S_k$ be fixed. Considering a_{k+1} as a variable digit, we have $x^2 - x = \left(a_{k+1}10^k + y\right)^2 - \left(a_{k+1}10^k + y\right) = (y^2 - y) + a_{k+1}10^k(2y-1) + a_{k+1}^2 10^{2k}$. Since $y^2 - y = 10^k z$ for an iteger z, it follows that $x^2 - x$ is divisible by 10^{k+1} if and only if $z + a_{k+1}(2y-1) \equiv 0 \pmod{10}$. Since $y \equiv 3 \pmod{10}$ is obviously impossible, the congruence has exactly one solution. Hence we obtain a one-to-one correspondence between the sets S_{k+1} and S_k for every $k \ge 1$. Therefore $s(2006) = s(1) = 3$, because $S_1 = \{1, 5, 6\}$.

Solution 2. Since $x^2 - x = x(x-1)$ and the numbers x and $x-1$ are relatively prime, one of them must be divisible by 2^{2006} and one of them (may be the same) must be divisible by 5^{2006}. Therefore, x must satisfy the following two conditions:

$$x \equiv 0 \text{ or } 1 \pmod{2^{2006}};$$

$$x \equiv 0 \text{ or } 1 \pmod{5^{2006}}.$$

Altogether we have 4 cases. The Chinese remainder theorem yields that in each case there is a unique solution among the numbers $0, 1, \dots, 10^{2006} - 1$. These four numbers are different because each two gives different residues modulo 2^{2006} or 5^{2006}. Moreover, one of the numbers is 0 which is not allowed.

Therefore there exist 3 solutions.

Problem 3. Let A be an $n \times n$-matrix with integer entries and $b_1, \dots, b_k$ be integers satisfying $\det A = b_1 \cdot \ldots \cdot b_k$. Prove that there exist $n \times n$-matrices $B_1, \dots, B_k$ with integer entries such that $A = B_1 \cdot \ldots \cdot B_k$ and $\det B_i = b_i$ for all $i = 1, \dots, k$.
(20 points)

Solution. By induction, it is enough to consider the case $m = 2$. Furthermore, we can multiply A with any integral matrix with determinant 1 from the right or from the left, without changing the problem. Hence we can assume A to be upper triangular.

Lemma. Let A be an integral upper triangular matrix, and let b, c be integers satisfying $\det A = bc$. Then there exist integral upper triangular matrices B, C such that $\det B = b$, $\det C = c$, $A = BC$.

Proof. The proof is done by induction on n, the case $n = 1$ being obvious. Assume the statement is true for $n-1$. Let A, b, c as in the statement of the lemma. Define B_{nn} to be the greatest common divisor of b and A_{nn}, and put $C_{nn} = \frac{A_{nn}}{B_{nn}}$. Since A_{nn} divides bc, C_{nn} divides $\frac{b}{B_{nn}}c$, which divides c. Hence C_{nn} divides c. Therefore, $b' = \frac{b}{B_{nn}}$ and $c' = \frac{c}{C_{nn}}$ are integers. Define A' to be the upper-left $(n-1) \times (n-1)$-submatrix of A; then $\det A' = b'c'$. By induction we can find the upper-left $(n-1) \times (n-1)$-part of B and C in such a way that $\det B = b$, $\det C = c$ and $A = BC$ holds on the upper-left $(n-1) \times (n-1)$-submatrix of A. It remains to define $B_{i,n}$ and $C_{i,n}$ such that $A = BC$ also holds for the (i, n)-th entry for all $i < n$.

First we check that B_{ii} and C_{nn} are relatively prime for all $i < n$. Since B_{ii} divides b', it is certainly enough to prove that b' and C_{nn} are relatively prime, i.e.

$$\gcd\left(\frac{b}{\gcd(b, A_{nn})}, \frac{A_{nn}}{\gcd(b, A_{nn})}\right) = 1,$$

which is obvious. Now we define $B_{j,n}$ and $C_{j,n}$ inductively: Suppose we have defined $B_{i,n}$ and $C_{i,n}$ for all $i = j+1, j+2, \ldots, n-1$. Then $B_{j,n}$ and $C_{j,n}$ have to satisfy

$$A_{j,n} = B_{j,j}C_{j,n} + B_{j,j+1}C_{j+1,n} + \cdots + B_{j,n}C_{n,n}$$

Since $B_{j,j}$ and $C_{n,n}$ are relatively prime, we can choose integers $C_{j,n}$ and $B_{j,n}$ such that this equation is satisfied. Doing this step by step for all $j = n-1, n-2, \ldots, 1$, we finally get B and C such that $A = BC$. □

Problem 4. Let f be a rational function (i.e. the quotient of two real polynomials) and suppose that $f(n)$ is an integer for infinitely many integers n. Prove that f is a polynomial.
(20 points)

Solution. Let S be an infinite set of integers such that rational function $f(x)$ is integral for all $x \in S$.

Suppose that $f(x) = p(x)/q(x)$ where p is a polynomial of degree k and q is a polynomial of degree n. Then p, q are solutions to the simultaneous equations $p(x) = q(x)f(x)$ for all $x \in S$ that are not roots of q. These are linear simultaneous equations in the coefficients of p, q with rational coefficients. Since they have a solution, they have a rational solution.

Thus there are polynomials p', q' with rational coefficients such that $p'(x) = q'(x)f(x)$ for all $x \in S$ that are not roots of q. Multiplying this with the previous equation, we see that $p'(x)q(x)f(x) = p(x)q'(x)f(x)$ for all $x \in S$ that are not roots of q. If x is not a root of p or q, then $f(x) \neq 0$, and hence $p'(x)q(x) = p(x)q'(x)$ for all $x \in S$ except for finitely many roots of p and q. Thus the two polynomials $p'q$ and pq' are equal for infinitely many choices of value. Thus $p'(x)q(x) = p(x)q'(x)$. Dividing by $q(x)q'(x)$, we see that $p'(x)/q'(x) = p(x)/q(x) = f(x)$. Thus $f(x)$ can be written as the quotient of two polynomials with rational coefficients. Multiplying up by some integer, it can be written as the quotient of two polynomials with integer coefficients.

Suppose $f(x) = p''(x)/q''(x)$ where p'' and q'' both have integer coefficients. Then by Euler's division algorithm for polynomials, there exist polynomials s and r, both of which have rational coefficients such that $p''(x) = q''(x)s(x) + r(x)$ and the degree of r is less than the degree of q''. Dividing by $q''(x)$, we get that $f(x) = s(x) + r(x)/q''(x)$. Now there exists an integer N such that $Ns(x)$ has integral coefficients. Then $Nf(x) - Ns(x)$ is an integer for all $x \in S$. However, this is equal to the rational function Nr/q'', which has a higher degree denominator than numerator, so tends to 0 as x tends to ∞. Thus for all sufficiently large $x \in S$, $Nf(x) - Ns(x) = 0$ and hence $r(x) = 0$. Thus r has infinitely many roots, and is 0. Thus $f(x) = s(x)$, so f is a polynomial.

Problem 5. Let $a, b, c, d, e > 0$ be real numbers such that $a^2 + b^2 + c^2 = d^2 + e^2$ and $a^4 + b^4 + c^4 = d^4 + e^4$. Compare the numbers $a^3 + b^3 + c^3$ and $d^3 + e^3$.
(20 points)

Solution. Without loss of generality $a \ge b \ge c$, $d \ge e$. Let $c^2 = e^2 + \Delta$, $\Delta \in \mathbb{R}$. Then $d^2 = a^2 + b^2 + \Delta$ and the second equation implies

$$a^4 + b^4 + (e^2 + \Delta)^2 = (a^2 + b^2 + \Delta)^2 + e^4, \quad \Delta = -\tfrac{a^2b^2}{a^2+b^2-e^2}.$$

(Here $a^2 + b^2 - e^2 \ge \frac{2}{3}(a^2 + b^2 + c^2) - \frac{1}{2}(d^2 + e^2) = \frac{1}{6}(d^2 + e^2) > 0$.)

Since $c^2 = e^2 - \frac{a^2b^2}{a^2+b^2-e^2} = \frac{(a^2-e^2)(e^2-b^2)}{a^2+b^2-e^2} > 0$ then $a > e > b$.

Therefore $d^2 = a^2 + b^2 - \frac{a^2b^2}{a^2+b^2-e^2} < a^2$ and $a > d \ge e > b \ge c$.

Consider a function $f(x) = a^x + b^x + c^x - d^x - e^x$, $x \in \mathbb{R}$. We shall prove that $f(x)$ has only two zeroes $x = 2$ and $x = 4$ and changes the sign at these points. Suppose the contrary. Then Rolle's theorem implies that $f'(x)$ has at least two distinct zeroes. Without loss of generality $a = 1$. Then $f'(x) = \ln b \cdot b^x + \ln c \cdot c^x - \ln d \cdot d^x - \ln e \cdot e^x$, $x \in \mathbb{R}$. If $f'(x_1) = f'(x_2) = 0$, $x_1 < x_2$, then

$$\ln b \cdot b^{x_i} + \ln c \cdot c^{x_i} = \ln d \cdot d^{x_i} + \ln e \cdot e^{x_i}, \quad i = 1, 2,$$

but since $1 > d \ge e > b \ge c$ we have

$$\frac{(-\ln b) \cdot b^{x_2} + (-\ln c) \cdot c^{x_2}}{(-\ln b) \cdot b^{x_1} + (-\ln c) \cdot c^{x_1}} \le b^{x_2 - x_1} < e^{x_2 - x_1} \le \frac{(-\ln d) \cdot d^{x_2} + (-\ln e) \cdot e^{x_2}}{(-\ln d) \cdot d^{x_1} + (-\ln e) \cdot e^{x_1}},$$

a contradiction. Therefore $f(x)$ has a constant sign at each of the intervals $(-\infty, 2)$, $(2, 4)$ and $(4, \infty)$. Since $f(0) = 1$ then $f(x) > 0$, $x \in (-\infty, 2) \bigcup (4, \infty)$ and $f(x) < 0$, $x \in (2, 4)$. In particular, $f(3) = a^3 + b^3 + c^3 - d^3 - e^3 < 0$.

Problem 6. Find all sequences $a_0, a_1, \ldots, a_n$ of real numbers where $n \ge 1$ and $a_n \ne 0$, for which the following statement is true:

If $f : \mathbb{R} \to \mathbb{R}$ is an n times differentiable function and $x_0 < x_1 < \ldots < x_n$ are real numbers such that $f(x_0) = f(x_1) = \ldots = f(x_n) = 0$ then there exists an $h \in (x_0, x_n)$ for which

$$a_0 f(h) + a_1 f'(h) + \ldots + a_n f^{(n)}(h) = 0.$$

(20 points)

Solution. Let $A(x) = a_0 + a_1 x + \ldots + a_n x^n$. We prove that sequence $a_0, \ldots, a_n$ satisfies the required property if and only if all zeros of polynomial $A(x)$ are real.

(a) Assume that all roots of $A(x)$ are real. Let us use the following notations. Let I be the identity operator on $\mathbb{R} \to \mathbb{R}$ functions and D be differentiation operator. For an arbitrary polynomial $P(x) = p_0 + p_1 x + \ldots + p_n x^n$, write $P(D) = p_0 I + p_1 D + p_2 D^2 + \ldots + p_n D^n$. Then the statement can written as $(A(D)f)(\xi) = 0$.

First prove the statement for $n = 1$. Consider the function

$$g(x) = e^{\frac{a_0}{a_1}x} f(x).$$

Since $g(x_0) = g(x_1) = 0$, by Rolle's theorem there exists a $\xi \in (x_0, x_1)$ for which

$$g'(\xi) = \frac{a_0}{a_1} e^{\frac{a_0}{a_1}\xi} f(\xi) + e^{\frac{a_0}{a_1}\xi} f'\xi) = \frac{e^{\frac{a_0}{a_1}\xi}}{a_1}(a_0 f(\xi) + a_1 f'(\xi)) = 0.$$

Now assume that $n > 1$ and the statement holds for $n-1$. Let $A(x) = (x-c)B(x)$ where c is a real root of polynomial A. By the $n = 1$ case, there exist $y_0 \in (x_0, x_1)$, $y_1 \in (x_1, x_2)$, $\ldots$, $y_{n-1} \in (x_{n-1}, x_n)$ such that $f'(y_j) - cf(y_j) = 0$ for all $j = 0, 1, \ldots, n-1$. Now apply the induction hypothesis for polynomial $B(x)$, function $g = f' - cf$ and points $y_0, \ldots, y_{n-1}$. The hypothesis says that there exists a $\xi \in (y_0, y_{n-1}) \subset (x_0, x_n)$ such that

$$(B(D)g)(\xi) = (B(D)(D - cI)f)(\xi) = (A(D)f)(\xi) = 0.$$

(b) Assume that $u + vi$ is a complex root of polynomial $A(x)$ such that $v \ne 0$. Consider the linear differential equation $a_n g^{(n)} + \ldots + a_1 g' + g = 0$. A solution of this equation is $g_1(x) = e^{ux} \sin vx$ which has infinitely many zeros.

Let k be the smallest index for which $a_k \ne 0$. Choose a small $\varepsilon > 0$ and set $f(x) = g_1(x) + \varepsilon x^k$. If ε is sufficiently small then g has the required number of roots but $a_0 f + a_1 f' + \ldots + a_n f^{(n)} = a_k \varepsilon \ne 0$ everywhere.