# IMC2007, Blagoevgrad, Bulgaria

## Day 1, August 5, 2007

**Problem 1.** Let $f$ be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer $k$. Prove that all coefficients of $f$ are divisible by 5.

**Solution 1.** Let $f(x) = ax^2 + bx + c$. Substituting $x = 0$, $x = 1$ and $x = -1$, we obtain that $5|f(0) = c$, $5|f(1) = (a+b+c)$ and $5|f(-1) = (a-b+c)$. Then $5|f(1)+f(-1)-2f(0) = 2a$ and $5|f(1)-f(-1) = 2b$. Therefore 5 divides $2a$, $2b$ and $c$ and the statement follows.

**Solution 2.** Consider $f(x)$ as a polynomial over the 5-element field (i.e. modulo 5). The polynomial has 5 roots while its degree is at most 2. Therefore $f \equiv 0 \pmod 5$ and all of its coefficients are divisible by 5.

**Problem 2.** Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose $n^2$ entries are precisely the numbers $1, 2, \dots, n^2$ ?

**Solution.** The minimal rank is 2 and the maximal rank is $n$. To prove this, we have to show that the rank can be 2 and $n$ but it cannot be 1.

(i) The rank is at least 2. Consider an arbitrary matrix $A = [a_{ij}]$ with entries $1, 2, \dots, n^2$ in some order. Since permuting rows or columns of a matrix does not change its rank, we can assume that $1 = a_{11} < a_{21} < \dots < a_{n1}$ and $a_{11} < a_{12} < \dots < a_{1n}$. Hence $a_{n1} \geq n$ and $a_{1n} \geq n$ and at least one of these inequalities is strict. Then $\det \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} < 1 \cdot n^2 - n \cdot n = 0$ so $\operatorname{rk}(A) \geq \operatorname{rk} \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} \geq 2$.

(ii) The rank can be 2. Let

$$T = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix}$$

The $i^{\text{th}}$ row is $(1, 2, \dots, n) + n(i-1) \cdot (1, 1, \dots, 1)$ so each row is in the two-dimensional subspace generated by the vectors $(1, 2, \dots, n)$ and $(1, 1, \dots, 1)$. We already proved that the rank is at least 2, so $\operatorname{rk}(T) = 2$.

(iii) The rank can be $n$, i.e. the matrix can be nonsingular. Put odd numbers into the diagonal, only even numbers above the diagonal and arrange the entries under the diagonal arbitrarily. Then the determinant of the matrix is odd, so the rank is complete.

**Problem 3.** Call a polynomial $P(x_1, \dots, x_k)$ *good* if there exist $2 \times 2$ real matrices $A_1, \dots, A_k$ such that

$$P(x_1, \dots, x_k) = \det \left( \sum_{i=1}^{k} x_i A_i \right).$$

Find all values of $k$ for which all homogeneous polynomials with $k$ variables of degree 2 are good.

(A polynomial is homogeneous if each term has the same total degree.)

**Solution.** The possible values for $k$ are 1 and 2.

If $k = 1$ then $P(x) = \alpha x^2$ and we can choose $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$.

If $k = 2$ then $P(x, y) = \alpha x^2 + \beta y^2 + \gamma xy$ and we can choose matrices $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & \beta \\ -1 & \gamma \end{pmatrix}$.

Now let $k \geq 3$. We show that the polynomial $P(x_1, \dots, x_k) = \sum_{i=0}^{k} x_i^2$ is not good. Suppose that $P(x_1, \dots, x_k) = \det \left( \sum_{i=0}^{k} x_i A_i \right)$. Since the first columns of $A_1, \dots, A_k$ are linearly dependent, the first column of some non-trivial linear combination $y_1A_1 + \ldots + y_kA_k$ is zero. Then $\det(y_1A_1 + \ldots + y_kA_k) = 0$ but $P(y_1, \ldots, y_k) \neq 0$, a contradiction.

**Problem 4.** Let $G$ be a finite group. For arbitrary sets $U, V, W \subset G$, denote by $N_{UVW}$ the number of triples $(x, y, z) \in U \times V \times W$ for which $xyz$ is the unity.

Suppose that $G$ is partitioned into three sets $A$, $B$ and $C$ (i.e. sets $A, B, C$ are pairwise disjoint and $G = A \cup B \cup C$). Prove that $N_{ABC} = N_{CBA}$.

**Solution.** We start with three preliminary observations.

Let $U, V$ be two arbitrary subsets of $G$. For each $x \in U$ and $y \in V$ there is a unique $z \in G$ for which $xyz = e$. Therefore,

$$N_{UVG} = |U \times V| = |U| \cdot |V|. \tag{1}$$

Second, the equation $xyz = e$ is equivalent to $yzx = e$ and $zxy = e$. For arbitrary sets $U, V, W \subset G$, this implies

$$\{(x,y,z) \in U \times V \times W :\ xyz = e\} = \{(x,y,z) \in U \times V \times W :\ yzx = e\} = \{(x,y,z) \in U \times V \times W :\ zxy = e\}$$

and therefore

$$N_{UVW} = N_{VWU} = N_{WUV}. \tag{2}$$

Third, if $U, V \subset G$ and $W_1, W_2, W_3$ are disjoint sets and $W = W_1 \cup W_2 \cup W_3$ then, for arbitrary $U, V \subset G$,

$$\{(x,y,z) \in U \times V \times W :\ xyz = e\} = \{(x,y,z) \in U \times V \times W_1 :\ xyz = e\} \cup$$

$$\cup \{(x,y,z) \in U \times V \times W_2 :\ xyz = e\} \cup \{(x,y,z) \in U \times V \times W_3 :\ xyz = e\}$$

so

$$N_{UVW} = N_{UVW_1} + N_{UVW_2} + N_{UVW_3}. \tag{3}$$

Applying these observations, the statement follows as

$$N_{ABC} = N_{ABG} - N_{ABA} - N_{ABB} = |A| \cdot |B| - N_{BAA} - N_{BAB} =$$

$$= N_{BAG} - N_{BAA} - N_{BAB} = N_{BAC} = N_{CBA}.$$

**Problem 5.** Let $n$ be a positive integer and $a_1, \ldots, a_n$ be arbitrary integers. Suppose that a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies $\sum_{i=1}^{n} f(k + a_i\ell) = 0$ whenever $k$ and $\ell$ are integers and $\ell \neq 0$. Prove that $f = 0$.

**Solution.** Let us define a subset $\mathcal{I}$ of the polynomial ring $\mathbb{R}[X]$ as follows:

$$\mathcal{I} = \Big\{ P(X) = \sum_{j=0}^{m} b_j X^j \ :\ \sum_{j=0}^{m} b_j f(k + j\ell) = 0 \quad \text{for all } k, \ell \in \mathbb{Z},\ \ell \neq 0 \Big\}.$$

This is a subspace of the real vector space $\mathbb{R}[X]$. Furthermore, $P(X) \in \mathcal{I}$ implies $X \cdot P(X) \in \mathcal{I}$. Hence, $\mathcal{I}$ is an ideal, and it is non-zero, because the polynomial $R(X) = \sum_{i=1}^{n} X^{a_i}$ belongs to $\mathcal{I}$. Thus, $\mathcal{I}$ is generated (as an ideal) by some non-zero polynomial $Q$.

If $Q$ is constant then the definition of $\mathcal{I}$ implies $f = 0$, so we can assume that $Q$ has a complex zero $c$. Again, by the definition of $\mathcal{I}$, the polynomial $Q(X^m)$ belongs to $\mathcal{I}$ for every natural number $m \geq 1$; hence $Q(X)$ divides $Q(X^m)$. This shows that all the complex numbers

$$c, c^2, c^3, c^4, \ldots$$

are roots of $Q$. Since $Q$ can have only finitely many roots, we must have $c^N = 1$ for some $N \geq 1$; in particular, $Q(1) = 0$, which implies $P(1) = 0$ for all $P \in \mathcal{I}$. This contradicts the fact that $R(X) = \sum_{i=1}^{n} X^{a_i} \in \mathcal{I}$, and we are done.

**Problem 6.** How many nonzero coefficients can a polynomial $P(z)$ have if its coefficients are integers and $|P(z)| \le 2$ for any complex number $z$ of unit length?

**Solution.** We show that the number of nonzero coefficients can be 0, 1 and 2. These values are possible, for example the polynomials $P_0(z) = 0$, $P_1(z) = 1$ and $P_2(z) = 1 + z$ satisfy the conditions and they have 0, 1 and 2 nonzero terms, respectively.

Now consider an arbitrary polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ satisfying the conditions and assume that it has at least two nonzero coefficients. Dividing the polynomial by a power of $z$ and optionally replacing $p(z)$ by $-p(z)$, we can achieve $a_0 > 0$ such that conditions are not changed and the number of nonzero terms is preserved. So, without loss of generality, we can assume that $a_0 > 0$.

Let $Q(z) = a_1 z + \ldots + a_{n-1} z^{n-1}$. Our goal is to show that $Q(z) = 0$.

Consider those complex numbers $w_0, w_1, \ldots, w_{n-1}$ on the unit circle for which $a_n w_k^n = |a_n|$; namely, let

$$w_k = \begin{cases} e^{2k\pi i/n} & \text{if } a_n > 0 \\ e^{(2k+1)\pi i/n} & \text{if } a_n < 0 \end{cases} \qquad (k = 0, 1, \ldots, n).$$

Notice that

$$\sum_{k=0}^{n-1} Q(w_k) = \sum_{k=0}^{n-1} Q(w_0 e^{2k\pi i/n}) = \sum_{j=1}^{n-1} a_j w_0^j \sum_{k=0}^{n-1} (e^{2j\pi i/n})^k = 0.$$

Taking the average of polynomial $P(z)$ at the points $w_k$, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} P(w_k) = \frac{1}{n}\sum_{k=0}^{n-1} \big(a_0 + Q(w_k) + a_n w_k^n\big) = a_0 + |a_n|$$

and

$$2 \ge \frac{1}{n}\sum_{k=0}^{n-1} \big|P(w_k)\big| \ge \left|\frac{1}{n}\sum_{k=0}^{n-1} P(w_k)\right| = a_0 + |a_n| \ge 2.$$

This obviously implies $a_0 = |a_n| = 1$ and $\big|P(w_k)\big| = \big|2 + Q(w_k)\big| = 2$ for all $k$. Therefore, all values of $Q(w_k)$ must lie on the circle $|2 + z| = 2$, while their sum is 0. This is possible only if $Q(w_k) = 0$ for all $k$. Then polynomial $Q(z)$ has at least $n$ distinct roots while its degree is at most $n - 1$. So $Q(z) = 0$ and $P(z) = a_0 + a_n z^n$ has only two nonzero coefficients.

**Remark.** From Parseval's formula (i.e. integrating $|P(z)|^2 = P(z)\overline{P(z)}$ on the unit circle) it can be obtained that

$$|a_0|^2 + \ldots + |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} \big|P(e^{it})\big|^2 \,\mathrm{d}t \le \frac{1}{2\pi}\int_0^{2\pi} 4 \,\mathrm{d}t = 4. \tag{4}$$

Hence, there cannot be more than four nonzero coefficients, and if there are more than one nonzero term, then their coefficients are $\pm 1$.

It is also easy to see that equality in (4) cannot hold two or more nonzero coefficients, so it is sufficient to consider only polynomials of the form $1 \pm x^m \pm x^n$. However, we do not know (yet :-)) any simpler argument for these cases than the proof above.