Problem 6. For a permutation $\sigma = (i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$ define $D(\sigma) = \sum_{k=1}^{n} |i_k - k|$. Let $Q(n, d)$ be the number of permutations σ of $(1, 2, ..., n)$ with $d = D(\sigma)$. Prove that $Q(n, d)$ is even for $d \geq 2n$.

Solution. Consider the $n \times n$ determinant

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-1} \\ x & 1 & \dots & x^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x^{n-1} & x^{n-2} & \dots & 1 \end{vmatrix}$$

where the ij-th entry is $x^{|i-j|}$. From the definition of the determinant we get

$$\Delta(x) = \sum_{(i_1,...,i_n) \in S_n} (-1)^{\mathrm{inv}(i_1,...,i_n)} x^{D(i_1,...,i_n)}$$

where S_n is the set of all permutations of $(1, 2, ..., n)$ and $\mathrm{inv}(i_1, ..., i_n)$ denotes the number of inversions in the sequence $(i_1, ..., i_n)$. So $Q(n, d)$ has the same parity as the coefficient of x^d in $\Delta(x)$.

It remains to evaluate $\Delta(x)$. In order to eliminate the entries below the diagonal, subtract the $(n-1)$-th row, multiplied by x, from the n-th row. Then subtract the $(n-2)$-th row, multiplied by x, from the $(n-1)$-th and so on. Finally, subtract the first row, multiplied by x, from the second row.

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ x & 1 & \dots & x^{n-3} & x^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & \dots & 1 & x \\ x^{n-1} & x^{n-2} & \dots & x & 1 \end{vmatrix} = \ldots = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ 0 & 1-x^2 & \dots & x^{n-3}-x^{n-1} & x^{n-2}-x^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-x^2 & x-x^3 \\ 0 & 0 & \dots & 0 & 1-x^2 \end{vmatrix} = (1-x^2)^{n-1}.$$

For $d \geq 2n$, the coefficient of x^d is 0 so $Q(n, d)$ is even.

IMC2008, Blagoevgrad, Bulgaria

Day 1, July 27, 2008

Problem 1. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for all reals x and y such that $x - y$ is rational.

Solution. We prove that $f(x) = ax + b$ where $a \in \mathbb{Q}$ and $b \in \mathbb{R}$. These functions obviously satify the conditions.

Suppose that a function $f(x)$ fulfills the required properties. For an arbitrary rational q, consider the function $g_q(x) = f(x+q) - f(x)$. This is a continuous function which attains only rational values, therefore g_q is constant.

Set $a = f(1) - f(0)$ and $b = f(0)$. Let n be an arbitrary positive integer and let $r = f(1/n) - f(0)$. Since $f(x + 1/n) - f(x) = f(1/n) - f(0) = r$ for all x, we have

$$f(k/n) - f(0) = (f(1/n) - f(0)) + (f(2/n) - f(1/n)) + \ldots + (f(k/n) - f((k-1)/n) = kr$$

and

$$f(-k/n) - f(0) = -(f(0) - f(-1/n)) - (f(-1/n) - f(-2/n)) - \ldots - (f(-(k-1)/n) - f(-k/n) = -kr$$

for $k \geq 1$. In the case $k = n$ we get $a = f(1) - f(0) = nr$, so $r = a/n$. Hence, $f(k/n) - f(0) = kr = ak/n$ and then $f(k/n) = a \cdot k/n + b$ for all integers k and $n > 0$.

So, we have $f(x) = ax + b$ for all rational x. Since the function f is continous and the rational numbers form a dense subset of $\mathbb{R}$, the same holds for all real x.

Problem 2. Denote by V the real vector space of all real polynomials in one variable, and let $P\colon V \to \mathbb{R}$ be a linear map. Suppose that for all $f, g \in V$ with $P(fg) = 0$ we have $P(f) = 0$ or $P(g) = 0$. Prove that there exist real numbers x_0, c such that $P(f) = c\,f(x_0)$ for all $f \in V$.

Solution. We can assume that $P \neq 0$.

Let $f \in V$ be such that $P(f) \neq 0$. Then $P(f^2) \neq 0$, and therefore $P(f^2) = aP(f)$ for some non-zero real a. Then $0 = P(f^2 - af) = P(f(f - a))$ implies $P(f - a) = 0$, so we get $P(a) \neq 0$. By rescaling, we can assume that $P(1) = 1$. Now $P(X + b) = 0$ for $b = -P(X)$. Replacing P by $\hat{P}$ given as

$$\hat{P}(f(X)) = P(f(X + b))$$

we can assume that $P(X) = 0$.

Now we are going to prove that $P(X^k) = 0$ for all $k \geq 1$. Suppose this is true for all $k < n$. We know that $P(X^n + e) = 0$ for $e = -P(X^n)$. From the induction hypothesis we get

$$P\big((X + e)(X + 1)^{n-1}\big) = P(X^n + e) = 0,$$

and therefore $P(X + e) = 0$ (since $P(X + 1) = 1 \neq 0$). Hence $e = 0$ and $P(X^n) = 0$, which completes the inductive step. From $P(1) = 1$ and $P(X^k) = 0$ for $k \geq 1$ we immediately get $P(f) = f(0)$ for all $f \in V$.

Problem 3. Let p be a polynomial with integer coefficients and let $a_1 < a_2 < \ldots < a_k$ be integers.

a) Prove that there exists $a \in \mathbb{Z}$ such that $p(a_i)$ divides $p(a)$ for all $i = 1, 2, \ldots, k$.

b) Does there exist an $a \in \mathbb{Z}$ such that the product $p(a_1) \cdot p(a_2) \cdot \ldots \cdot p(a_k)$ divides $p(a)$?

Solution. The theorem is obvious if $p(a_i) = 0$ for some i, so assume that all $p(a_i)$ are nonzero and pairwise different.

There exist numbers s, t such that $s|p(a_1)$, $t|p(a_2)$, $st = \operatorname{lcm}(p(a_1), p(a_2))$ and $\gcd(s,t) = 1$.

As s, t are relatively prime numbers, there exist $m, n \in \mathbb{Z}$ such that $a_1 + sn = a_2 + tm =: b_2$. Obviously $s|p(a_1 + sn) - p(a_1)$ and $t|p(a_2 + tm) - p(a_2)$, so $st|p(b_2)$.

Similarly one obtains b_3 such that $p(a_3)|p(b_3)$ and $p(b_2)|p(b_3)$ thus also $p(a_1)|p(b_3)$ and $p(a_2)|p(b_3)$.

Reasoning inductively we obtain the existence of $a = b_k$ as required.

The polynomial $p(x) = 2x^2 + 2$ shows that the second part of the problem is not true, as $p(0) = 2$, $p(1) = 4$ but no value of $p(a)$ is divisible by 8 for integer a.

Remark. One can assume that the $p(a_i)$ are nonzero and ask for a such that $p(a)$ is a nonzero multiple of all $p(a_i)$. In the solution above, it can happen that $p(a) = 0$. But every number $p(a + np(a_1)p(a_2)\ldots p(a_k))$ is also divisible by every $p(a_i)$, since the polynomial is nonzero, there exists n such that $p(a + np(a_1)p(a_2)\ldots p(a_k))$ satisfies the modified thesis.

Problem 4. We say a triple (a_1, a_2, a_3) of nonnegative reals is *better* than another triple (b_1, b_2, b_3) if **two out of the three** following inequalities $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$ are satisfied. We call a triple (x, y, z) *special* if x, y, z are nonnegative and $x + y + z = 1$. Find all natural numbers n for which there is a set S of n *special* triples such that for any given *special* triple we can find at least one *better* triple in S.

Solution. The answer is $n \geqslant 4$.

Consider the following set of special triples:

$$\left(0, \frac{8}{15}, \frac{7}{15}\right), \qquad \left(\frac{2}{5}, 0, \frac{3}{5}\right), \qquad \left(\frac{3}{5}, \frac{2}{5}, 0\right), \qquad \left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right).$$

We will prove that any special triple (x, y, z) is worse than one of these (triple a is worse than triple b if triple b is better than triple a). We suppose that some special triple (x, y, z) is actually not worse than the first three of the triples from the given set, derive some conditions on x, y, z and prove that, under these conditions, (x, y, z) is worse than the fourth triple from the set.

Triple (x, y, z) is not worse than $\left(0, \frac{8}{15}, \frac{7}{15}\right)$ means that $y \geqslant \frac{8}{15}$ or $z \geqslant \frac{7}{15}$. Triple (x, y, z) is not worse than $\left(\frac{2}{5}, 0, \frac{3}{5}\right)$ — $x \geqslant \frac{2}{5}$ or $z \geqslant \frac{3}{5}$. Triple (x, y, z) is not worse than $\left(\frac{3}{5}, \frac{2}{5}, 0\right)$ — $x \geqslant \frac{3}{5}$ or $y \geqslant \frac{2}{5}$. Since $x + y + z = 1$, then it is impossible that all inequalities $x \geqslant \frac{2}{5}$, $y \geqslant \frac{2}{5}$ and $z \geqslant \frac{7}{15}$ are true. Suppose that $x < \frac{2}{5}$, then $y \geqslant \frac{2}{5}$ and $z \geqslant \frac{3}{5}$. Using $x + y + z = 1$ and $x \geqslant 0$ we get $x = 0, y = \frac{2}{5}, z = \frac{3}{5}$. We obtain the triple $\left(0, \frac{2}{5}, \frac{3}{5}\right)$ which is worse than $\left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right)$. Suppose that $y < \frac{2}{5}$, then $x \geqslant \frac{3}{5}$ and $z \geqslant \frac{7}{15}$ and this is a contradiction to the admissibility of (x, y, z). Suppose that $z < \frac{7}{15}$, then $x \geqslant \frac{2}{5}$ and $y \geqslant \frac{8}{15}$. We get (by admissibility, again) that $z \leqslant \frac{1}{15}$ and $y \leqslant \frac{3}{5}$. The last inequalities imply that $\left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right)$ is better than (x, y, z).

We will prove that for any given set of three special triples one can find a special triple which is not worse than any triple from the set. Suppose we have a set S of three special triples

$$(x_1, y_1, z_1), \qquad (x_2, y_2, z_2), \qquad (x_3, y_3, z_3).$$

Denote $a(S) = min(x_1, x_2, x_3), b(S) = min(y_1, y_2, y_3), c(S) = min(z_1, z_2, z_3)$. It is easy to check that S_1:

$$\left(\frac{x_1 - a}{1 - a - b - c}, \frac{y_1 - b}{1 - a - b - c}, \frac{z_1 - c}{1 - a - b - c}\right)$$
$$\left(\frac{x_2 - a}{1 - a - b - c}, \frac{y_2 - b}{1 - a - b - c}, \frac{z_2 - c}{1 - a - b - c}\right)$$
$$\left(\frac{x_3 - a}{1 - a - b - c}, \frac{y_3 - b}{1 - a - b - c}, \frac{z_3 - c}{1 - a - b - c}\right)$$

is a set of three special triples also (we may suppose that $a + b + c < 1$, because otherwise all three triples are equal and our statement is trivial).

If there is a special triple (x, y, z) which is not worse than any triple from S_1, then the triple

$$((1 - a - b - c)x + a, (1 - a - b - c)y + b, (1 - a - b - c)z + c)$$

is special and not worse than any triple from S. We also have $a(S_1) = b(S_1) = c(S_1) = 0$, so we may suppose that the same holds for our starting set S.

Suppose that one element of S has two entries equal to 0.

Note that one of the two remaining triples from S is not worse than the other. This triple is also not worse than all triples from S because any special triple is not worse than itself and the triple with two zeroes.

So we have $a = b = c = 0$ but we may suppose that all triples from S contain at most one zero. By transposing triples and elements in triples (elements in all triples must be transposed simultaneously) we may achieve the following situation $x_1 = y_2 = z_3 = 0$ and $x_2 \geqslant x_3$. If $z_2 \geqslant z_1$, then the second triple $(x_2, 0, z_2)$ is not worse than the other two triples from S. So we may assume that $z_1 \geqslant z_2$. If $y_1 \geqslant y_3$, then the first triple is not worse than the second and the third and we assume $y_3 \geqslant y_1$. Consider the three pairs of numbers x_2, y_1; z_1, x_3; y_3, z_2. The sum of all these numbers is three and consequently the sum of the numbers in one of the pairs is less than or equal to one. If it is the first pair then the triple $(x_2, 1 - x_2, 0)$ is not worse than all triples from S, for the second we may take $(1 - z_1, 0, z_1)$ and for the third — $(0, y_3, 1 - y_3)$. So we found a desirable special triple for any given S.

Problem 5. Does there exist a finite group G with a normal subgroup H such that $|\operatorname{Aut} H| > |\operatorname{Aut} G|$?

Solution. Yes. Let H be the commutative group $H = \mathbb{F}_2^3$, where $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ is the field with two elements. The group of automorphisms of H is the general linear group $\mathrm{GL}_3\mathbb{F}_2$; it has

$$(8 - 1) \cdot (8 - 2) \cdot (8 - 4) = 7 \cdot 6 \cdot 4 = 168$$

elements. One of them is the shift operator $\phi : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$.

Now let $T = \{a^0, a^1, a^2\}$ be a group of order 3 (written multiplicatively); it acts on H by $\tau(a) = \phi$. Let G be the semidirect product $G = H \rtimes_\tau T$. In other words, G is the group of 24 elements

$$G = \{ba^i : \ b \in H, \, i \in (\mathbb{Z}/3\mathbb{Z})\}, \quad ab = \phi(b)a.$$

G has one element e of order 1 and seven elements b, $b \in H$, $b \neq e$ of order 2.

If $g = ba$, we find that $g^2 = baba = b\phi(b)a^2 \neq e$, and that

$$g^3 = b\phi(b)a^2ba = b\phi(b)a\phi(b)a^2 = b\phi(b)\phi^2(b)a^3 = \psi(b),$$

where the homomorphism $\psi : H \to H$ is defined as $\psi : (x_1, x_2, x_3) \mapsto (x_1 + x_2 + x_3)(1, 1, 1)$. It is clear that $g^3 = \psi(b) = e$ for 4 elements $b \in H$, while $g^6 = \psi^2(b) = e$ for all $b \in H$.

We see that G has 8 elements of order 3, namely ba and ba^2 with $b \in \operatorname{Ker}\psi$, and 8 elements of order 6, namely ba and ba^2 with $b \notin \operatorname{Ker}\psi$. That accounts for orders of all elements of G.

Let $b_0 \in H \backslash \operatorname{Ker}\psi$ be arbitrary; it is easy to see that G is generated by b_0 and a. As every automorphism of G is fully determined by its action on b_0 and a, it follows that G has no more than

$$7 \cdot 8 = 56$$

automorphisms.

Remark. G and H can be equivalently presented as subgroups of S_6, namely as $H = \langle(12), (34), (56)\rangle$ and $G = \langle(135)(246), (12)\rangle$.