IMC2010, Blagoevgrad, Bulgaria

Day 1, July 26, 2010

Problem 1. Let $0 < a < b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2}dx \ge e^{-a^2} - e^{-b^2}.$$

Solution 1. Let $f(x) = \int_0^x (t^2+1)e^{-t^2}dt$ and let $g(x) = -e^{-x^2}$; both functions are increasing.

By Cauchy's Mean Value Theorem, there exists a real number $x \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)} = \frac{(x^2+1)e^{-x^2}}{2xe^{-x^2}} = \frac{1}{2}\left(x + \frac{1}{x}\right) \ge \sqrt{x \cdot \frac{1}{x}} = 1.$$

Then

$$\int_a^b (x^2+1)e^{-x^2}dx = f(b) - f(a) \ge g(b) - g(a) = e^{-a^2} - e^{-b^2}.$$

Solution 2.

$$\int_a^b (x^2+1)e^{-x^2}dx \ge \int_a^b 2xe^{-x^2}dx = \left[-e^{-x^2}\right]_a^b = e^{-a^2} - e^{-b^2}.$$

Problem 2. Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{5\cdot6\cdot7\cdot8} + \cdots .$$

Solution 1. Let

$$F(x) = \sum_{k=0}^{\infty} \frac{x^{4k+4}}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

This power series converges for $|x| \le 1$ and our goal is to compute $F(1)$.

Differentiating 4 times, we get

$$F^{(IV)}(x) = \sum_{k=0}^{\infty} x^{4k} = \frac{1}{1-x^4}.$$

Since $F(0) = F'(0) = F''(0) = F'''(0) = 0$ and F is continuous at $1-0$ by Abel's continuity theorem, integrating 4 times we get

$$F'''(y)=F'''(0)+\int_0^y F^{(IV)}(x)\ \mathrm{d}x=\int_0^y \frac{\mathrm{d}x}{1-x^4}=\frac12\arctan y+\frac14\log(1+y)-\frac14\log(1-y)\,,$$

$$F''(z)=F''(0)+\int_0^z F'''(y)\ \mathrm{d}x=\int_0^z\left(\frac12\arctan y+\frac14\log(1+y)-\frac14\log(1-y)\right)\ \mathrm{d}y=$$

$$=\frac12\left(z\arctan z-\int_0^z\frac{y}{1+y^2}\ \mathrm{d}y\right)+\frac14\left((1+z)\log(1+z)-\int_0^z\ \mathrm{d}y\right)+\frac14\left((1-z)\log(1-z)+\int_0^z\ \mathrm{d}y\right)=$$

$$=\frac12 z\arctan z-\frac14\log(1+z^2)+\frac14(1+z)\log(1+z)+\frac14(1-z)\log(1-z)\,,$$

$$F'(t)=\int_0^t\left(\frac12 z\arctan z-\frac14\log(1+z^2)+\frac14(1+z)\log(1+z)+\frac14(1-z)\log(1-z)\right)\ \mathrm{d}t=$$

$$=\frac14\Big((1+t^2)\arctan t-t\Big)-\frac14\Big(t\log(1+t^2)-2t+2\arctan t\Big)+$$

$$+\frac18\left((1+t)^2\log(1+t)-t-\frac12t^2\right)-\frac18\left((1-t)^2\log(1-t)+t-\frac12t^2\right)=$$

$$=\frac14(-1+t^2)\arctan t-\frac14t\log(1+t^2)+\frac18(1+t)^2\log(1+t)-\frac18(1-t)^2\log(1-t)\,,$$

$$F(1)=\int_0^1\left(\frac14(-1+t^2)\arctan t-\frac14t\log(1+t^2)+\frac18(1+t)^2\log(1+t)-\frac18(1-t)^2\log(1-t)\right)\ \mathrm{d}t=$$

$$=\left[\frac{-3t+t^3}{12}\arctan t+\frac{1-3t^2}{24}\log(1+t^2)+\frac{(1+t)^3}{24}\log(1+t)+\frac{(1-t)^3}{24}\log(1-t)\right]_0^1=\frac{\ln 2}{4}-\frac{\pi}{24}.$$

Remark. The computation can be shorter if we change the order of integrations.

$$F(1)=\int_{t=0}^1\int_{z=0}^t\int_{y=0}^z\int_{x=0}^y\frac{1}{1-x^4}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\ \mathrm{d}t=\int_{x=0}^1\frac{1}{1-x^4}\int_{y=x}^1\int_{z=y}^1\int_{t=z}^1\ \mathrm{d}t\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}z=$$

$$=\int_{x=0}^1\frac{1}{1-x^4}\left(\frac16\int_{y=x}^1\int_{z=x}^1\int_{t=x}^1\ \mathrm{d}t\ \mathrm{d}z\ \mathrm{d}y\right)\ \mathrm{d}x=\int_0^1\frac{1}{1-x^4}\cdot\frac{(1-x)^3}{6}\ \mathrm{d}x=$$

$$=\left[-\frac16\arctan x-\frac1{12}\log(1+x^2)+\frac13\log(1+x)\right]_0^1=\frac{\ln 2}{4}-\frac{\pi}{24}.$$

Solution 2. Let

$$A_m=\sum_{k=0}^m\frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}=\sum_{k=0}^m\left(\frac16\cdot\frac{1}{4k+1}-\frac12\cdot\frac{1}{4k+2}+\frac12\cdot\frac{1}{4k+3}-\frac16\cdot\frac{1}{4k+4}\right),$$

$$B_m=\sum_{k=0}^m\left(\frac{1}{4k+1}-\frac{1}{4k+3}\right),$$

$$C_m=\sum_{k=0}^m\left(\frac{1}{4k+1}-\frac{1}{4k+2}+\frac{1}{4k+3}-\frac{1}{4k+4}\right)\quad\text{and}$$

$$D_m=\sum_{k=0}^m\left(\frac{1}{4k+2}-\frac{1}{4k+4}\right).$$

It is easy check that

$$A_m=\frac13C_m-\frac16B_m-\frac16D_m.$$

Therefore,

$$\lim A_m=\lim\frac{2C_m-B_m-D_m}{6}=\frac{2\ln 2-\frac{\pi}{4}-\frac12\ln 2}{6}=\frac14\ln 2-\frac{\pi}{24}.$$

Problem 3. Define the sequence $x_1, x_2, \ldots$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}.$$

Solution. Let $y_n = x_n^2$. Then $y_{n+1} = (y_n - 2)^2$ and $y_{n+1} - 4 = y_n(y_n - 4)$. Since $y_2 = 9 > 5$, we have $y_3 = (y_2 - 2)^2 > 5$ and inductively $y_n > 5, n \geq 2$. Hence, $y_{n+1} - y_n = y_n^2 - 5y_n + 4 > 4$ for all $n \geq 2$, so $y_n \to \infty$.

By $y_{n+1} - 4 = y_n(y_n - 4)$,

$$\begin{aligned}\left(\frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}\right)^2 &= \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1}} \\ &= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1} - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_{n-1}}{y_n - 4} = \cdots \\ &= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{1}{y_1 - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \to 1.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}} = 1.$$

Problem 4. Let a, b be two integers and suppose that n is a positive integer for which the set

$$\mathbb{Z} \setminus \left\{ax^n + by^n \mid x, y \in \mathbb{Z}\right\}$$

is finite. Prove that $n = 1$.

Solution. Assume that $n > 1$. Notice that n may be replaced by any prime divisor p of n. Moreover, a and b should be coprime, otherwise the numbers not divisible by the greatest common divisor of a, b cannot be represented as $ax^n + by^n$.

If $p = 2$, then the number of the form $ax^2 + by^2$ takes not all possible remainders modulo 8. If, say, b is even, then ax^2 takes at most three different remainders modulo 8, by^2 takes at most two, hence $ax^2 + by^2$ takes at most $3 \times 2 = 6$ different remainders. If both a and b are odd, then $ax^2 + by^2 \equiv x^2 \pm y^2 \pmod 4$; the expression $x^2 + y^2$ does not take the remainder 3 modulo 4 and $x^2 - y^2$ does not take the remainder 2 modulo 4.

Consider the case when $p \geq 3$. The pth powers take exactly p different remainders modulo p^2. Indeed, $(x + kp)^p$ and x^p have the same remainder modulo p^2, and all numbers 0^p, 1^p, ..., $(p-1)^p$ are different even modulo p. So, $ax^p + by^p$ take at most p^2 different remainders modulo p^2. If it takes strictly less then p^2 different remainders modulo p^2, we get infinitely many non-representable numbers. If it takes exactly p^2 remainders, then $ax^p + by^p$ is divisible by p^2 only if both x and y are divisible by p. Hence if $ax^p + by^p$ is divisible by p^2, it is also divisible by p^p. Again we get infinitely many non-representable numbers, for example the numbers congruent to p^2 modulo p^3 are non-representable.

Problem 5. Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that

$$1 + 2abc \geq a^2 + b^2 + c^2 .$$

Prove that

$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$$

for all positive integers n.

Solution 1. Consider the symmetric matrix

$$A = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}.$$

By the constraint we have $\det A \geq 0$ and $\det \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \geq 0$. Hence A is positive semidefinite, and $A = B^2$ for some symmetric real matrix B.

Let the rows of B be x, y, z. Then $|x| = |y| = |z| = 1$, $a = x \cdot y$, $b = y \cdot z$ and $c = z \cdot x$, where $|x|$ and $x \cdot y$ denote the Euclidean norm and scalar product. Denote by $X = \otimes^n x$, $Y = \otimes^n y$, $Z = \otimes^n z$ the nth tensor powers, which belong to $\mathbb{R}^{3^n}$. Then $|X| = |Y| = |Z| = 1$, $X \cdot Y = a^n$, $Y \cdot Z = b^n$ and $Z \cdot X = c^n$. So, the matrix $\begin{pmatrix} 1 & a^n & b^n \\ a^n & 1 & c^n \\ b^n & c^n & 1 \end{pmatrix}$, being the Gram matrix of three vectors in $\mathbb{R}^{3^n}$, is positive semidefinite, and its determinant, $1 + 2(abc)^n - a^{2n} - b^{2n} - c^{2n}$ is non-negative.

Solution 2. The constraint can be written as

$$(a - bc)^2 \leq (1 - b^2)(1 - c^2). \tag{1}$$

By the Cauchy-Schwarz inequality,

$$\left(a^{n-1} + a^{n-2}bc + \ldots + b^{n-1}c^{n-1}\right)^2 \leq \left(|a|^{n-1} + |a|^{n-2}|bc| + \ldots + |bc|^{n-1}\right)^2 \leq$$
$$\leq \left(1 + |bc| + \ldots + |bc|^{n-1}\right)^2 \leq \left(1 + |b|^2 + \ldots + |b|^{2(n-1)}\right)\left(1 + |c|^2 + \ldots + |c|^{2(n-1)}\right)$$

Multiplying by (1), we get

$$(a - bc)^2(a^{n-1} + a^{n-2}bc + \ldots + b^{n-1}c^{n-1})^2 \leq$$
$$\left((1 - b^2)\left(1 + |b|^2 + \ldots + |b|^{2(n-1)}\right)\right)\left((1 - c^2)\left(1 + |c|^2 + \ldots + |c|^{2(n-1)}\right)\right),$$
$$(a^n - b^n c^n)^2 \leq (1 - b^n)(1 - c^n),$$
$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + b^{2n}.$$