IMC2011, Blagoevgrad, Bulgaria

Day 2, July 31, 2011

Problem 1. Let $(a_n)_{n=0}^{\infty}$ be a sequence with $\frac{1}{2} < a_n < 1$ for all $n \ge 0$. Define the sequence $(x_n)_{n=0}^{\infty}$ by

$$x_0 = a_0, \qquad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} \quad (n \ge 0).$$

What are the possible values of $\lim_{n\to\infty} x_n$? Can such a sequence diverge?

Johnson Olaleru, Lagos

Solution 1. We prove by induction that

$$0 < 1 - x_n < \frac{1}{2^{n+1}}.$$

Then we will have $(1 - x_n) \to 0$ and therefore $x_n \to 1$.

The case $n = 0$ is true since $\frac{1}{2} < x_0 = a_0 < 1$.

Supposing that the induction hypothesis holds for n, from the recurrence relation we get

$$1 - x_{n+1} = 1 - \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} = \frac{1 - a_{n+1}}{1 + a_{n+1}x_n}(1 - x_n).$$

By

$$0 < \frac{1 - a_{n+1}}{1 + a_{n+1}x_n} < \frac{1 - \frac{1}{2}}{1 + 0} = \frac{1}{2}$$

we obtain

$$0 < 1 - x_{n+1} < \frac{1}{2}(1 - x_n) < \frac{1}{2} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}.$$

Hence, the sequence converges in all cases and $x_n \to 1$.

Solution 2. As is well-known,

$$\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

for all real numbers u and v.

Setting $u_n = \operatorname{ar\,tanh} a_n$ we have $x_n = \tanh(u_0 + u_1 + \cdots + u_n)$. Then $u_0 + u_1 + \cdots + u_n > (n+1)\operatorname{ar\,tanh} \frac{1}{2}$ and $\lim_{n\to\infty} x_n = \lim_{u\to\infty} \tanh u = 1$.

Remark. If the condition $a_n \in (\frac{1}{2}, 1)$ is replaced by $a_n \in (0, 1)$ then the sequence remains increasing and bounded, but the limit can be less than 1.

Problem 2. An alien race has three genders: male, female, and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual — if x likes y, then y likes x.

The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

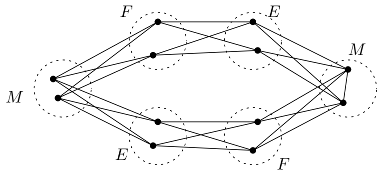
a) Show that if n is even and $k = \frac{n}{2}$, then it might be impossible to create even one married triple.

b) Show that if $k \ge \frac{3n}{4}$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.

Fedor Duzhin and Nick Gravin, Singapore

Solution. (a) Let M be the set of males, F the set of females, and E the set of emales. Consider the (tripartite) graph G with vertices $M \cup F \cup E$ and edges for likes. A 3-cycle is then a possible family. We'll call G the *graph of likes.*

First, let $k = \frac{n}{2}$. Then n has to be even and we need to construct a graph of likes with no 3-cycles. We'll do the following: divide each of the sets M, F, and E into two equal parts and draw all edges between two parts as shown below:



Clearly, there is no 3-cycle.

(b) First divide the the expedition into male-emale-female triples arbitrarily. Let the *unhappiness* of such a subdivision be the number of pairs of aliens that belong to the same triple but don't like each other. We shall show that if unhappiness is positive, then the unhappiness can be decreased by a simple operation. It will follow that after several steps the unhappiness will be reduced to zero, which will lead to the happy marriage of everybody.

Assume that we have an emale which doesn't like at least one member of its triple (the other cases are similar). We perform the following operation: we swap this emale with another emale, so that each of these two emales will like the members of their new triples. Thus the unhappiness related to this emales will decrease, and the other pairs that contribute to the unhappiness remain unchanged, therefore the unhappiness will be decreased.

So, it remains to prove that such an operation is always possible. Enumerate the triples with $1, 2, \dots, n$ and denote by E_i, F_i, M_i the emale, female, and male members of the ith triple, respectively. Without loss of generality we may assume that E_1 doesn't like either F_1 or M_1 or both. We have to find an index $i > 1$ such that E_i likes the couple F_1, M_1 and E_1 likes the couple F_i, M_i; then we can swap E_1 and E_i.

There are at most $n/4$ indices i for which E_1 dislikes F_i and at most $n/4$ indices for which E_1 dislikes M_i, so there are no more than $n/2$ indices i for which E_1 dislikes someone from the couple M_i, F_i, and the set of these undesirable indexes includes 1. Similarly, there are no more than $n/2$ indices such that either M_1 or F_1 dislikes E_i. Since both undesirable sets of indices have at most $n/2$ elements and both contain 1, their union doesn't cover all indices, so we have some i which satisfies all conditions. Therefore we can always perform the operation that decreases unhappiness.

Solution 2 (for part b). Suppose that $k \geq \frac{3n}{4}$ and let's show that it's possible to marry all of the colonists. First, we'll prove that there exists a perfect matching between M and F. We need to check the condition of Hall's marriage theorem. In other words, for $A \subset M$, let $B \subset F$ be the set of all vertices of F adjacent to at least one vertex of A. Then we need to show that $|A| \leq |B|$. Let us assume the contrary, that is $|A| > |B|$. Clearly, $|B| \geq k$ if A is not empty. Let's consider any $f \in F \setminus B$. Then f is not adjacent to any vertex in A, therefore, f has degree in M not more than $n - |A| < n - |B| \leq n - k \leq \frac{n}{4}$, a contradiction.

Let's now construct a new bipartite graph, say H. The set of its vertices is $P \cup E$, where P is the set of pairs male–female from the perfect matching we just found. We will have an edge from $(m, f) = p \in P$ to $e \in E$ for each 3-cycle (m, f, e) of the graph G, where $(m, f) \in P$ and $e \in E$. Notice that the degree of each vertex of P in H is then at least $2k - n$.

What remains is to show that H satisfies the condition of Hall's marriage theorem and hence has a perfect matching. Assume, on the contrary, that the following happens. There is $A \subset P$ and $B \subset E$ such that $|A| = l$, $|B| < l$, and B is the set of all vertices of E adjacent to at least one vertex of A. Since the degree of each vertex of P is at least $2k - n$, we have $2k - n \leq |B| < l$. On the other hand, let $e \in E \setminus B$. Then for each pair $(m, f) = p \in P$, at most one of the pairs (e, m) and (e, f) is joined by an edge and hence the degree of e in G is at most $|M \setminus A| + |F \setminus A| + |A| = 2(n - l) + l = 2n - l$. But the degree of any vertex of G is $2k$ and thus we get $2k \leq 2n - l$, that is, $l \leq 2n - 2k$.

Finally, $2k - n < l \leq 2n - 2k$ implies that $k < \frac{3n}{4}$. This contradiction concludes the solution.

Problem 3. Determine the value of

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \cdot \ln\left(1 + \frac{1}{2n}\right) \cdot \ln\left(1 + \frac{1}{2n+1}\right).$$

Gerhard Woeginger, Utrecht

Solution. Define $f(n) = \ln(\frac{n+1}{n})$ for $n \geq 1$, and observe that $f(2n)+f(2n+1) = f(n)$. The well-known inequality $\ln(1+x) \leq x$ implies $f(n) \leq 1/n$. Furthermore introduce

$$g(n) \;=\; \sum_{k=n}^{2n-1} f^3(k) \;<\; n\, f^3(n) \;\leq\; 1/n^2.$$

Then

$$\begin{aligned}
g(n) - g(n+1) &= f^3(n) - f^3(2n) - f^3(2n+1) \\
&= (f(2n) + f(2n+1))^3 - f^3(2n) - f^3(2n+1) \\
&= 3\,(f(2n) + f(2n+1))\; f(2n)\, f(2n+1) \\
&= 3\, f(n)\, f(2n)\, f(2n+1),
\end{aligned}$$

therefore

$$\sum_{n=1}^{N} f(n)\, f(2n)\, f(2n+1) \;=\; \frac{1}{3} \sum_{n=1}^{N} g(n) - g(n+1) \;=\; \frac{1}{3}\,(g(1) - g(N+1))\,.$$

Since $g(N+1) \to 0$ as $N \to \infty$, the value of the considered sum hence is

$$\sum_{n=1}^{\infty} f(n)\, f(2n)\, f(2n+1) \;=\; \frac{1}{3}\, g(1) \;=\; \frac{1}{3}\, \ln^3(2).$$

Problem 4. Let $f(x)$ be a polynomial with real coefficients of degree n. Suppose that $\dfrac{f(k) - f(m)}{k - m}$ is an integer for all integers $0 \leq k < m \leq n$. Prove that $a - b$ divides $f(a) - f(b)$ for all pairs of distinct integers a and b.

Fedor Petrov, St. Petersburg

Solution 1. We need the following

Lemma. Denote the least common multiple of $1, 2, \ldots, k$ by $L(k)$, and define

$$h_k(x) = L(k) \cdot \binom{x}{k} \qquad (k = 1, 2, \ldots).$$

Then the polynomial $h_k(x)$ satisfies the condition, i.e. $a - b$ divides $h_k(a) - h_k(b)$ for all pairs of distinct integers a, b.

Proof. It is known that

$$\binom{a}{k} = \sum_{j=0}^{k} \binom{a-b}{j} \binom{b}{k-j}.$$

(This formula can be proved by comparing the coefficient of x^k in $(1+x)^a$ and $(1+x)^{a-b}(1+x)^b$.) From here we get

$$h_k(a) - h_k(b) = L(K) \left(\binom{a}{k} - \binom{b}{k} \right) = L(K) \sum_{j=1}^{k} \binom{a-b}{j} \binom{b}{k-j} = (a-b) \sum_{j=1}^{k} \frac{L(k)}{j} \binom{a-b-1}{j-1} \binom{b}{k-j}.$$

On the right-hand side all fractions $\frac{L(k)}{j}$ are integers, so the right-hand side is a multiple of (a, b). The lemma is proved.

Expand the polynomial f in the basis $1, \binom{x}{1}, \binom{x}{2}, \ldots$ as

$$f(x) = A_0 + A_1 \binom{x}{1} + A_2 \binom{x}{2} + \cdots + A_n \binom{x}{n}. \tag{1}$$

We prove by induction on j that A_j is a multiple of $L(j)$ for $1 \leq j \leq n$. (In particular, A_j is an integer for $j \geq 1$.) Assume that $L(j)$ divides A_j for $1 \leq j \leq m-1$. Substituting m and some $k \in \{0, 1, \ldots, m-1\}$ in (1),

$$\frac{f(m) - f(k)}{m - k} = \sum_{j=1}^{m-1} \frac{A_j}{L(j)} \cdot \frac{h_j(m) - h_j(k)}{m - k} + \frac{A_m}{m - k}.$$

Since all other terms are integers, the last term $\frac{A_m}{m-k}$ is also an integer. This holds for all $0 \le k < m$, so A_m is an integer that is divisible by $L(m)$.

Hence, A_j is a multiple of $L(j)$ for every $1 \le j \le n$. By the lemma this implies the problem statement.

Solution 2. The statement of the problem follows immediately from the following claim, applied to the polynomial $g(x,y) = \frac{f(x)-f(y)}{x-y}$.

Claim. Let $g(x,y)$ be a real polynomial of two variables with total degree less than n. Suppose that $g(k,m)$ is an integer whenever $0 \le k < m \le n$ are integers. Then $g(k,m)$ is a integer for every pair k,m of integers.

Proof. Apply induction on n. If $n=1$ then g is a constant. This constant can be read from $g(0,1)$ which is an integer, so the claim is true.

Now suppose that $n \ge 2$ and the claim holds for $n-1$. Consider the polynomials

$$g_1(x,y) = g(x+1,y+1) - g(x,y+1) \quad \text{and} \quad g_2(x,y) = g(x,y+1) - g(x,y). \tag{1}$$

For every pair $0 \le k < m \le n-1$ of integers, the numbers $g(k,m)$, $g(k,m+1)$ and $g(k+1,m+1)$ are all integers, so $g_1(k,m)$ and $g_2(k,m)$ are integers, too. Moreover, in (1) the maximal degree terms of g cancel out, so $\deg g_1, \deg g_2 < \deg g$. Hence, we can apply the induction hypothesis to the polynomials g_1 and g_2 and we thus have $g_1(k,m), g_2(k,m) \in \mathbb{Z}$ for all $k,m \in \mathbb{Z}$.

In view of (1), for all $k,m \in \mathbb{Z}$, we have that

(a) $g(0,1) \in \mathbb{Z}$;

(b) $g(k,m) \in \mathbb{Z}$ if and only if $g(k+1,m+1) \in \mathbb{Z}$;

(c) $g(k,m) \in \mathbb{Z}$ if and only if $g(k,m+1) \in \mathbb{Z}$.

For arbitrary integers k,m, apply *(b)* $|k|$ times then apply *(c)* $|m-k-1|$ times as

$$g(k,m) \in \mathbb{Z} \Leftrightarrow \ldots \Leftrightarrow g(0,m-k) \in \mathbb{Z} \Leftrightarrow \ldots \Leftrightarrow g(0,1) \in \mathbb{Z}.$$

Hence, $g(k,m) \in \mathbb{Z}$. The claim has been proved.

Problem 5. Let $F = A_0A_1\ldots A_n$ be a convex polygon in the plane. Define for all $1 \le k \le n-1$ the operation f_k which replaces F with a new polygon

$$f_k(F) = A_0 \ldots A_{k-1}A'_kA_{k+1}\ldots A_n\ ,$$

where A'_k is the point symmetric to A_k with respect to the perpendicular bisector of $A_{k-1}A_{k+1}$. Prove that $(f_1 \circ f_2 \circ \ldots \circ f_{n-1})^n(F) = F$. We suppose that all operations are well-defined on the polygons, to which they are applied, i.e. results are convex polygons again. (A_0, $A_1, \ldots, A_n$ are the vertices of F in consecutive order.)

Mikhail Khristoforov, St. Petersburg

Solution. The operations f_i are rational maps on the $2(n-1)$-dimensional phase space of coordinates of the vertices $A_1, \ldots, A_{n-1}$. To show that $(f_1 \circ f_2 \circ \ldots \circ f_{n-1})^n$ is the identity, it is sufficient to verify this on some open set. For example, we can choose a neighborhood of the regular polygon, then all intermediate polygons in the proof will be convex.

Consider the operations f_i. Notice that (i) $f_i \circ f_i = id$ and (ii) $f_i \circ f_j = f_j \circ f_i$ for $|i-j| \ge 2$. We also show that (iii) $(f_i \circ f_{i+1})^3 = id$ for $1 \le i \le n-1$.

The operations f_i and f_{i+1} change the order of side lengths by interchanging two consecutive sides; after performing $(f_i \circ f_{i+1})^3$, the side lengths are in the original order. Moreover, the sums of opposite angles in the convex quadrilateral $A_{i-1}A_iA_{i+1}A_{i+2}$ are preserved in all operations. These quantities uniquely determine the quadrilateral, because with fixed sides, both angles $\angle A_1A_2A_3$ and $\angle A_1A_4A_3$ decrease when A_1A_3 increases. Hence, property (iii) is proved.

In the symmetric group S_n, the transpositions $\sigma_i = (i, i+1)$, which from a generator system, satisfy the same properties (i–iii). It is well-known that S_n is the maximal group with $n-1$ generators, satisfying (i–iii). In S_n we have $(\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_{n-1})^n = (1,2,3,\ldots,n)^n = id$, so this implies $(f_1 \circ f_2 \circ \ldots \circ f_{n-1})^n = id$.