ON FINITE HOMOGENEOUS METRIC SPACES

V.N. Berestovskii[1] and Yu.G. Nikonorov[2]

[1]Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences & Novosibirsk State University, Russia, vberestov@inbox.ru
[2]Southern Mathematical Institute of the Vladikavkaz Scientific Center of the Russian Academy of Sciences, Russia, nikonorov2006@mail.ru

We discuss the class of finite homogeneous metric spaces and some of its important subclasses that have natural definitions in terms of metrics and well-studied analogues in the class of Riemannian manifolds [1,3,4]. Recall that a metric space (M,d) is *homogeneous* if its full isometry group acts transitively on M. One can consider finite groups supplied with some left-invariant metrics as examples of finite homogeneous metric spaces. Let us consider some other useful definitions. Recall that a map between metric spaces is called a submetry if it maps closed balls of radius r around a point onto closed balls of the same radius around the image point [2].

A finite homogeneous metric space (M,d) is called *normal homogeneous* if there is a transitive isometry group Γ of (M,d) and a bi-invariant metric σ on Γ such that the canonical projection $\pi:(\Gamma,\sigma)\to(M,d)$ is a submetry.

A finite homogeneous metric space (M,d) is called *generalized normal homogeneous* if for every points $x,y\in M$ there is an isometry f (that is called a δ-translation) of the space (M,d) such that $f(x)=y$ and $d(x,f(x))\geq d(z,f(z))$ for all $z\in M$.

A finite homogeneous metric space (M,d) is called *strongly generalized normal homogeneous* if for every points $x,y\in M$, $x\neq y$, there is an isometry f of the space (M,d) such that f has no fixed point, $f(x)=y$, and $d(x,f(x))\geq d(z,f(z))$ for all $z\in M$.

A finite homogeneous metric space (M,d) is called *Clifford–Wolf homogeneous* (shortly, CW-homogeneous) if for every points $x,y\in M$ there is an isometry f (that is called a CW-translation) of the space (M,d) such that $f(x)=y$ and $d(x,f(x))=d(z,f(z))$ for any point $z\in M$.

Let us introduce the following notations for metric spaces: FGBM, FGLM, FCWHS, FSGNHS, FGNHS, FNHS, FHS denote respectively the classes of finite groups with bi-invariant metrics, of finite groups with left-invariant metrics, finite CW-homogeneous spaces, finite strongly generalized normal homogeneous spaces, finite generalized normal homogeneous spaces, finite normal homogeneous spaces, and finite homogeneous spaces. Our main result is the following

Theorem 1. *The following inclusions and equality are fulfilled:*

$$FGBM\subset FCWHS\subset FSGNHS\subset FGNHS=FNHS\subset FHS,$$

$$FGBM\subset FGLM\subset FHS.$$

Moreover, all the above inclusions are strict.

Examples of corresponding spaces are built, some of which are sets of vertices of special convex polytopes in Euclidean spaces. In particular, we proved

Theorem 2. *Let (M,d) be the set of vertices of a regular polyhedron (Platonic solid) in $\mathbb{E}^3$ with the induced metric. Then $(M,d)\in FGBM$ for the tetrahedron and for the cube; $(M,d)\in FCWHS$ and $(M,d)\notin FGBM$ for the octahedron; $(M,d)\in FHS$ and $(M,d)\notin FNHS$ for the dodecahedron; $(M,d)\in FGNHS=FNHS$ and $(M,d)\notin FSGNHS$ for the icosahedron.*

We also obtained the description of multidimensional regular polytopes. At first, we recall the following well known result.

Proposition 1 [6]. *Any finite subgroup of $G\subset S^3$ endowed with the metric d, induced from the Euclidean metric on $\mathbb{H}=\mathbb{R}^4$, is a Clifford–Wolf homogeneous metric space, and multiplying by elements of G, both on the left and on the right, are CW-translations.*

Our main result for multidimensional regular polytopes is the following one.

Theorem 3. *The set of vertices M of every regular polytopes of dimension $n \geq 4$ with the metric d, induced from the Euclidean metric on $\mathbb{E}^n$, is Clifford–Wolf homogeneous, with the exception of the 120-cell in $\mathbb{E}^4$, the set of vertices of which is not even normal homogeneous.*

The classes for the sets of vertices of semiregular polytopes in $\mathbb{E}^3$ are determined too.

We also give the description of the classes of metric spaces under consideration in terms of graph theory, with the help of which examples of finite metric spaces with unusual properties are constructed. We consider in details one construction related to Kneser graphs, see e. g. Chapter 7 in [5].

Let us consider some $k, n \in \mathbb{N}$, $1 \leq k \leq n/2$. The Kneser graph $KG_{n,k} = (V, E)$ is a graph whose vertices are all k-element subsets of the set $\mathbb{Z}_n$ (or any other n-element set), and the edges are pairs of such subsets with empty intersection, i. e. $V = \{A \subset \mathbb{Z}_n \,|\, |A| = k\}$ and $E = \{\{A, B\} \mid A, B \in V,\ A \cap B = \emptyset\}$. It is clear that $|V| = C_n^k = \frac{n!}{k!(n-k)!}$. Note that for $k = 1$ we obtain the complete graph K_n, for $k = n/2$ the Kneser graph is a perfect matching, $KG_{5,2}$ is the Petersen graph.

Now, for a given Kneser graph $KG_{n,k} = (V, E)$ with $k \geq 2$, we define the finite metric space $(M = V, d)$. Consider some positive numbers $\alpha_1 \neq \alpha_2$ such that $\alpha_1 < 2\alpha_2 < 4\alpha_1$ and define the metric $d = d_{n,k,\alpha_1,\alpha_2}$ as follows: $d(A, B) = \alpha_1$ for $A \cap B = \emptyset$ and $d(A, B) = \alpha_2$ for $A \cap B \neq \emptyset$ $(A, B \in V,\ A \neq B)$.

It is not difficult to prove that for $k = n/2$ we get Clifford–Wolf homogeneous metric spaces $(M = V, d)$. In what follows we suppose that $2 \leq k < n/2$, in particular, $n \geq 5$.

It is well known that the authomorphism group of the graph $KG_{n,k}$ for $k < n/2$ coincides with the group $S(n)$ of all permutations of the set $\mathbb{Z}_n$, see e. g. Corollary 7.8.2 in [5].

Therefore, $S(n)$ is the full isometry group of the metric space $(M = V, d = d_{n,k,\alpha_1,\alpha_2})$: every permutation $\psi \in S(n)$ generates the isometry $i_\psi : V \to V$, acting by the formula

$$i_\psi(\{a_1, a_2, \ldots, a_k\}) = \{\psi(a_1), \psi(a_2), \ldots, \psi(a_k)\}.$$

In particular, all such spaces are homogeneous. We proved the following result.

Theorem 4. *The following assertions are fulfilled.*
1) *The metric space $(M = V, d = d_{n,k,\alpha_1,\alpha_2})$ is normal homogeneous if and only if $\alpha_1 > \alpha_2$.*
2) *The metric space $(M = V, d = d_{n,k,\alpha_1,\alpha_2})$, $n/2 > k \geq 2$, is not Clifford–Wolf homogeneous.*
3) *The metric space $(M = V, d = d_{n,k,\alpha_1,\alpha_2})$, $\alpha_1 > \alpha_2$, is strongly generalized normal homogeneous for $(n, k) = (7, 3)$ and for $(n, k) = (3m, m+1)$, where $m \geq 3$ and $m \not\equiv -1 (\mathrm{mod}\, 3)$.*

Funally, we discuss several unsolved problems.

References

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