The path homology theory

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A *path complex* on a finite set V is a collection of paths $v = (v_0, v_1, \dots, v_i)$ on V such that if a path v belongs to P then a truncated path, that is obtained from v by removing either the first or the last point, is also in P (see [1]). The main motivation for considering path complexes comes from the digraph theory. A digraph G is given by a pair (V, E) where V is the set of vertices and E is a set of arrow $a \to b$. Any digraph G defines naturally a path complex $P(G)$ where allowed paths are those that go along arrows.

Any path complex P allows to define a chain complex with an appropriate boundary operator that leads to the notion of homology groups $H_*(P)$ (see [1 – 7]. Thus, in particular, we obtain a homology theory for the category of digraphs [5, 6].

Moreover, any finite simplicial complex S defines a finite digraph G_S as follows (see [2]). The set of vertices of G_S coincides with the set of all simplexes from S, and there is an arrow $s \to t$ if and only if

$$s \supset t \text{ and } \dim s = \dim t + 1.$$

The simlicial homology groups $H_*(S)$ and the path homology groups $H_*(P(G_S))$ are isomorphic. Thus, the notion of a path homology provides an alternative viewpoint for the classical results about simplicial homology.

The path homology theory is compatible with the homotopy theory of digraphs, and these homology groups have good functorial properties with respect to graph-theoretical operations (see [3, 5, 6]). The notion of path complex give a new elementary proof of a representation of simplicial homology as a Hochschild homology (see [4]). There is a generalization of path homology groups to the category of quivers (see [7]).

References

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